Business PreCalculus MATH 1643 Section 004, Spring 2014

Lesson 27: Exponential and Logarithmic Equations

Definition 1. **Exponential Equations:** *An equation containing terms of the form a^x with $a > 0$ and $a \neq 1$ is called an* ***exponential equation****.*

Recall that in Lesson 23 we have learned how to solve an exponential equation if both sides of the equation can be expressed as a power of the same base. **However**, not all exponential equations can be written this way. In this lesson, we will learn how to solve any exponential and logarithmic equation.

Definition 2. **The one-to-one Property of Exponential and Logarithmic Functions:** *The one-to-one property of exponential and logarithmic functions states that*

1. *If $a^u = a^v$, then $u = v$.*
2. *If $\log_a u = \log_a v$, then $u = v$.*

Example 1. *Solve each equation.*

a. $9^x = 27^{x+1}$.

b. $\log_5(2x - 1) = \log_5(x + 2)$

Solution:

a.

$$9^x = 27^{x+1}$$
$$3^{2x} = 3^{3(x+1)}.$$

Then the one-to-one property of exponential functions implies that

$$2x = 3(x + 1),$$

which gives $x = -3$.

b.

$$\log_5(2x - 1) = \log_5(x + 2)$$
$$2x - 1 = x + 2 \qquad \textit{one-to-one property}$$
$$x = 3.$$

Definition 3. **Solving an Exponential Equation with Different Bases:** *To solve exponential equations with different bases:*

1. *Isolate the exponential expression.*
2. *Take the common or natural logarithm.*

3. Use the power rule, $\log_a M^r = r \log_a M$.

4. Solve for the variable.

Example 2. *Solve the equation $5.2^x - 7 = 10$.*

Solution: *Following the steps above:*

$$\begin{aligned}
5.2^x - 7 &= 10 \\
5.2^x &= 10 + 7 \\
2^x &= \frac{17}{5} \\
\ln 2^x &= \ln \frac{17}{5} \\
x \ln 2 &= \ln 17 - \ln 5 \\
x &= \frac{\ln 17 - \ln 5}{\ln 2}.
\end{aligned}$$

Example 3. *Solve the equation $2.3^{x-1} = 5^{x+1}$.*

Solution:

$$\begin{aligned}
2.3^{x-1} &= 5^{x+1} \\
\ln(2.3^{x-1}) &= \ln 5^{x+1} \\
\ln 2 + \ln 3^{x-1} &= \ln 5^{x+1} \\
\ln 2 + (x-1)\ln 3 &= (x+1)\ln 5 \\
\ln 2 + x\ln 3 - \ln 3 &= x \ln 5 + \ln 5 \\
x \ln 3 - x \ln 5 &= \ln 5 + \ln 3 - \ln 2 \\
x(\ln 3 - \ln 5) &= \ln 5 + \ln 3 - \ln 2 \\
x &= \frac{\ln 5 + \ln 3 - \ln 2}{\ln 3 - \ln 5}.
\end{aligned}$$

Definition 4. Logarithmic Equations: *An equation containing terms of the form $\log_a x$ is called a **logarithmic equation**.*

Example 4. *Solve the equation $3 + \log(2x + 5) = 2$.*

Solution:

$$\begin{aligned}
3 + \log(2x+5) &= 2 \\
\log(2x+5) &= -1 \\
2x + 5 &= 10^{-1} = \frac{1}{10} \\
2x &= \frac{1}{10} - 5 = \frac{-49}{10} \\
x &= \frac{-49}{20}.
\end{aligned}$$

Example 5. *Solve the equation* $\log_4 x + \log_4(x+1) = \log_4(x-1) + \log_4 6$.

Solution:

$$\begin{aligned} &\log_4 x + \log_4(x+1) = \log_4(x-1) + \log_4 6 \\ &\log_4[x(x+1)] = \log_4[6(x-1)] \\ &x(x+1) = 6(x-1) \quad \textit{one-to-one property} \\ &x^2 + x = 6x - 6 \\ &x^2 - 5x + 6 = 0 \\ &(x-2)(x-3) = 0 \end{aligned}$$

Hence, either $x = 2$ *or* $x = 3$. *The last step is to check these numbers in the original equation.*

CHECK $x = 2$***:***

$$\begin{aligned} \log_4 2 + \log_4(2+1) &= \log_4(2-1) + \log_4 6 \\ \log_4 2 + \log_4 3 &= \log_4 1 + \log_4 6 \\ \log_4(2.3) &= 0 + \log_4 6. \quad \textit{YES.} \end{aligned}$$

It is an exercise to check that $x = 3$ *is a solution as well.*

Remark 1. ***It is necessary to check the answers when solving logarithmic equations.***